Journal of Combinatorial Theory, Series A 220 (2026) 106132

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

journal homepage: www.elsevier.com/locate/jcta

Eulerian-type polynomials over Stirling permutations and box sorting algorithm

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A R T I C L E   I N F O

Article history:
Received 25 June 2025
Received in revised form 20 November 2025
Accepted 22 November 2025
Available online xxxx

Keywords:
Stirling permutations
Signed permutations
Proper ascent-plateaux
Box sorting algorithm
Standard Young tableaux

A B S T R A C T

It is well known that ascents, descents and plateaux are equidistributed over the set of classical Stirling permutations. Their common enumerative polynomials are the second-order Eulerian polynomials, which have been extensively studied by many researchers. This paper is divided into three parts. The first part gives a convolution formula for the second-order Eulerian polynomials, which simplifies a result of Gessel. As an application, a determinantal expression for the second-order Eulerian polynomial is obtained. We then investigate a convolution formula of the trivariate second-order Eulerian polynomials. Among other things, by introducing three new statistics: proper ascent-plateau, improper ascent-plateau and trace, we discover that a six-variable enumerative polynomial over restricted Stirling permutations equals a six-variable Eulerian-type polynomial over signed permutations. By special parametrizations, we make use of Stirling permutations to give a unified interpretation of the (p,q)-Eulerian polynomials and derangement polynomials of types A and B. The third part presents a box sorting algorithm which leads to a bijection between the terms in the expansion of $(cD)^n c$ and ordered weak set partitions, where c is a smooth function in the

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https://doi.org/10.1016/j.jcta.2025.106132

indeterminate x and D is the derivative with respect to x. Using a map from ordered weak set partitions to standard Young tableaux, we find an expansion of $(cD)^n c$ in terms of standard Young tableaux. Combining this with context-free grammars, we provide three new interpretations of the second-order Eulerian polynomials.

Contents

1. Introduction

Many polynomials can be generated by successive differentiations of a given base function, see [29, Section 4.3] for a survey. Here we provide two such polynomials. The Eulerian polynomials $A_n(x)$ first appeared in the following series summation or successive differentiation:

$$\sum_{k=0}^{\infty} k^n x^k = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{1-x} = \frac{A_n(x)}{(1-x)^{n+1}}. \tag{1}$$

The *second-order Eulerian polynomials* $C_n(x)$ can be defined by

$$\sum_{k=0}^{\infty} \left\{ {n+k \atop k} \right\} x^k = \left(\frac{x}{1-x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{x}{1-x} = \frac{C_n(x)}{(1-x)^{2n+1}}, \tag{2}$$

where $\left\{ {n \atop k} \right\}$ is the *Stirling number of the second kind*, i.e., the number of partitions of $[n] := \{1, 2, \ldots, n\}$ into exactly k blocks, see [9,11]. The polynomials $A_n(x)$ and $C_n(x)$ share several similar properties, including recursions [21,23,33], real-rootedness [25,35] and combinatorial expansions [14,41]. For example, we have

$$A_n(x) = nxA_{n-1}(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_{n-1}(x),\ A_0(x) = 1;$$
$$C_{n+1}(x) = (2n+1)xC_n(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}C_n(x),\ C_0(x) = 1. \tag{3}$$

This paper is motivated by the following problem.

Problem 1. Whether there exists a function $f_n(x_1, x_2, \ldots, x_n)$ such that

$$C_n(x) = f_n\left(A_1(x), A_2(x), \ldots, A_n(x)\right).$$

For $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$, let $\mathbf{n} = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ be a multiset, where the element i appears m_i times. A *multipermutation* of $\mathbf{n}$ is a sequence of its elements. For any word over $\mathbf{n}$, we say that the *reduced form* of w, written as $\mathrm{red}\,(w)$, is equal to the word obtained by replacing each of the occurrences of the i-th smallest number in w with the number i. Denote by $\mathfrak{S}_{\mathbf{n}}$ the set of multipermutations of $\mathbf{n}$. We say that a multipermutation σ of $\mathbf{n}$ is a *Stirling permutation* if for each i, $1 \leqslant i \leqslant n$, all letters occurring between the two occurrences of i are at least i. Denote by $\mathcal{Q}_{\mathbf{n}}$ the set of Stirling permutations of $\mathbf{n}$. When $m_1 = \cdots = m_n = 1$, the set $\mathcal{Q}_{\mathbf{n}}$ reduces to the symmetric group $\mathfrak{S}_n$, i.e., the set of permutations of $[n]$. When $m_1 = m_2 = \cdots = m_n = 2$, the set $\mathcal{Q}_{\mathbf{n}}$ reduces to the set $\mathcal{Q}_n$ (the set of classical Stirling permutations), which is defined by Gessel-Stanley [23]. Except where explicitly stated, we always assume that all Stirling permutations belong to $\mathcal{Q}_n$. For example, $\mathcal{Q}_1 = \{11\}$ and $\mathcal{Q}_2 = \{1122, 1221, 2211\}$.

For $\sigma \in \mathcal{Q}_{\mathbf{n}}$, any entry σ_i is called an *ascent* (resp. *descent*, *plateau*) if $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i > \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$), where $i \in \{0, 1, 2, \ldots, m_1 + m_2 + \cdots + m_n\}$ and we set $\sigma_0 = \sigma_{m_1+m_2+\cdots+m_n+1} = 0$. Let $\mathrm{asc}\,(\sigma)$ (resp. $\mathrm{des}\,(\sigma)$, $\mathrm{plat}\,(\sigma)$) be the number of ascents (resp. descents, plateaux) of σ. The *Eulerian polynomials* can also be defined by

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}\,(\pi)} = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k,$$

where $\left\langle {n \atop k} \right\rangle$ are known as the *Eulerian numbers*. It is well known that $A_n(x)$ is symmetric, see [38] for instance. When $\sigma \in \mathcal{Q}_n$, we always set $\sigma_0 = \sigma_{2n+1} = 0$. Gessel-Stanley [23] found that

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)}.$$

The *trivariate second-order Eulerian polynomials* are defined by

$$C_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{des}\,(\sigma)} z^{\mathrm{plat}\,(\sigma)}.$$

The first few $C_n(x, y, z)$ are listed as follows:

$$C_1(x,y,z) = xyz,\ C_2(x,y,z) = xyz(yz+xy+xz),$$
$$C_3(x,y,z) = xyz(x^2y^2+x^2z^2+y^2z^2+4xy^2z+4x^2yz+4xyz^2).$$

Set $C_0(x,y,z) = x$. Dumont [19, p. 317] discovered that

$$C_{n+1}(x,y,z) = xyz\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)C_n(x,y,z), \tag{4}$$

which implies that the polynomials $C_n(x,y,z)$ are symmetric in the variables x, y and z. It was independently discovered by Bóna [4] that $C_n(x) = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)}$. In recent years, there has been considerable study on Stirling permutations and their variants, see [8,15,22,25,30,33,34,36,38,39].

Note that each nonempty Stirling permutation $\sigma \in \mathcal{Q}_n$ can be represented as $\sigma'1\sigma''1\sigma'''$, where σ', σ'' and σ''' are all Stirling permutations (may be empty). Clearly, the descent number of σ is the sum of the descent numbers of σ', σ'' and σ''', unless σ''' is empty in which case σ has an additional descent. Motivated by this observation, Gessel [24] found that

$$\frac{\mathrm{d}}{\mathrm{d}z}C(x;z) = C^2(x;z)(C(x;z)+x-1),$$

where $C(x;z) = \sum_{n=0}^{\infty} C_n(x)\frac{z^n}{n!}$. Extracting the coefficient of $\frac{z^n}{n!}$, one can easily deduce that

$$C_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}C_k(x)C_{n-k}(x)+\sum_{k=0}^{n-1}\binom{n}{k}\left(\sum_{j=0}^{k}\binom{k}{j}C_j(x)C_{k-j}(x)\right)C_{n-k}(x). \tag{5}$$

Our point of departure is the following problem.

Problem 2. Whether there is a simplified version of (5)?

A *partition* of n is a weakly decreasing sequence of nonnegative integers: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where $\sum_{i=1}^{\ell}\lambda_i = n$. Each λ_i is called a *part* of λ. If λ is a partition of n, then we write $\lambda \vdash n$. We denote by m_i the number of parts equal to i. By using the multiplicities, we also denote λ by $(1^{m_1}2^{m_2}\cdots n^{m_n})$. The *length* of λ, denoted $\ell(\lambda)$, is the maximum subscript j such that $\lambda_j > 0$. The *Ferrers diagram* of λ is a graphical representation of λ with λ_i left justified boxes in its i-th row. For a Ferrers diagram $\lambda \vdash n$ (we will often identify a partition with its Ferrers diagram), a *standard Young tableau* (SYT, for short) of shape λ is a filling of the n boxes of λ with the integers in $[n]$ such that each of those integers is used exactly once, and all rows and columns are increasing (from left to right, and from bottom to top, respectively), and we number its rows starting from the bottom and going above. See [1] for instance.

Let SYT_λ be the set of standard Young tableaux of shape λ. Set $\mathrm{SYT}\,(n) = \bigcup_{\lambda\vdash n}\mathrm{SYT}_\lambda$. The *descent set* for $T\in\mathrm{SYT}\,(n)$ is defined by

$$\mathrm{Des}(T)=\{i\mid i+1 \text{ appears in an upper row of } T \text{ than } i, \text{ where } 1\leqslant i\leqslant n-1\}.$$

Denote by $\#V$ the cardinality of a set V. Let $\mathrm{des}\,(T)=\#\,\mathrm{Des}(T)$. The Robinson-Schensted correspondence is a bijection from permutations to pairs of standard Young tableaux of the same shape. This correspondence and its generalization, the Robinson-Schensted-Knuth correspondence, have become centerpieces of enumerative and algebraic combinatorics due to their many applications. Let $f^\lambda=\#\,\mathrm{SYT}_\lambda$. An application of the Robinson-Schensted correspondence is given as follows:

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{des}\,(\pi)}=\sum_{\lambda\vdash n}f^\lambda\sum_{T\in\mathrm{SYT}_\lambda}x^{\mathrm{des}\,(T)+1}, \tag{6}$$

which has been extended to skew shapes. See [1,28] for instance.

This paper is organized as follows. In the next section, we present a result concerning Problem 2. In Section 3, we investigate a convolution formula for the trivariate second-order Eulerian polynomials $C_n(x,y,z)$. A deep connection between signed permutations in the hyperoctahedral group and Stirling permutations is established. In particular, we introduce three new statistics on Stirling permutations: proper ascent-plateau, improper ascent-plateau and trace. A special case of Corollary 18 says that

$$\sum_{\sigma\in\mathcal{Q}_{n+1}^{(1)}}s^{\mathrm{impap}(\sigma)}t^{\mathrm{bk}_2(\sigma)}q^{\mathrm{tr}(\sigma)}=2^n\sum_{\pi\in\mathfrak{S}_n}\left(\frac{t+s}{2}\right)^{\mathrm{fix}\,(\pi)}q^{\mathrm{cyc}\,(\pi)},$$

where $\mathcal{Q}_n^{(1)}$ is the set of Stirling permutations over the multiset $\{1,2,2,3,3,\ldots,n,n\}$. In Section 4, using standard Young tableaux, we provide certain combinatorial expansions of the Eulerian polynomials of types A and B as well as the second-order Eulerian polynomials. In Theorem 28, we give an affirmative answer to Problem 1, i.e.,

$$C_n(x)=\sum_{T\in\mathrm{SYT}\,(n)}\prod_{i=1}^n\sigma_i(T)A_i(x)^{w_i(T)},$$

where $w_i(T)$ is the number of rows in the standard Young tableau T with i elements and $\prod_{i=1}^n\sigma_i(T)$ is the g-index of T introduced in [26]. See Table 2 for an illustration.

2. Convolution formulas for the second-order Eulerian polynomials

Since des and plat are equidistributed over Stirling permutations in $\mathcal{Q}_n$, it is natural to count Stirling permutations by the plateau statistic plat. We now present a variant of (5).

Theorem 3. *For $n \geqslant 1$, we have*

$$C_n(x) = nxC_{n-1}(x) + \sum_{r=1}^{n-1} \binom{n}{n-r+1} C_{n-r}(x)C_{r-1}(x).$$

Proof. Let σ be a Stirling permutation in $\mathcal{Q}_n$. Consider a decomposition of σ: $i\sigma' i\sigma''$, where σ' and σ'' are both Stirling permutations (may be empty). We distinguish two cases:

(i) If σ begins with the plateau ii, i.e. $\sigma' = \emptyset$, then there are n choices for i. When we count the number of plateaux of σ, this case gives the term $nxC_{n-1}(x)$.

(ii) When $\sigma' \neq \emptyset$, choose a multiset

$$\{a_1, a_1, a_2, a_2, \ldots, a_{n-r+1}, a_{n-r+1}\}$$

in $\binom{n}{n-r+1}$ ways, where $1 \leqslant i = a_1 < a_2 < \cdots < a_{n-r+1} \leqslant n$. Let S_{n-r} be the set of Stirling permutations over $\{a_2, a_2, a_3, a_3, \ldots, a_{n-r+1}, a_{n-r+1}\}$, and let $\sigma' \in S_{n-r}$. If we count the number of plateaux of σ', then S_{n-r} contributes to the term $C_{n-r}(x)$. Let $\overline{S}_{r-1}$ be the set of Stirling permutations over $\{1, 1, 2, 2, \ldots, n, n\} \setminus \{a_1, a_1, a_2, a_2, \ldots, a_{n-r+1}, a_{n-r+1}\}$. Then $\mathrm{red}\,(\sigma'') \in \mathcal{Q}_{r-1}$ and $\overline{S}_{r-1}$ contributes to the term $C_{r-1}(x)$.

The aforementioned two cases exhaust all the possibilities, and this completes the proof. $\square$

Let $a_n = 2^n C_n(1/2)$. It follows from Theorem 3 that

$$a_n = na_{n-1} + 2\sum_{r=1}^{n-1} \binom{n}{n-r+1} a_{n-r}a_{r-1}, \ a_0 = a_1 = 1.$$

It should be noted that a_n counts series-reduced rooted trees with $n+1$ labeled leaves [44, A000311]. The first few a_n are $1, 4, 26, 236, 2752$. Very recently, Bitonti-Deb-Sokal [2] obtained a continued fraction expression of a_n.

Lemma 4 *([35]). Let $F(x) = a(x)f(x) + b(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x)$, where $a(x)$ and $b(x)$ are two real polynomials. Both $f(x)$ and $F(x)$ have only positive leading coefficients, and $\deg F = \deg f$ or $\deg f + 1$. If $f(x)$ has only real zeros and $b(r) \leqslant 0$ whenever $f(r) = 0$, then $F(x)$ has only real zeros.*

Combining (3) and Lemma 4, one can obtain that the zeros of $C_{n-1}(x)$ are real and simple, which separate those of $C_n(x)$ for $n \geqslant 2$. Comparing (3) with Theorem 3, we find that

$$\sum_{r=1}^{n-1}\binom{n}{n-r+1}C_{n-r}(x)C_{r-1}(x)=(n-1)xC_{n-1}(x)+x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}C_{n-1}(x),$$

Note that

$$\begin{aligned}&nC_{n-1}(x)+\sum_{r=1}^{n-1}\binom{n}{n-r+1}C_{n-r}(x)C_{r-1}(x)\\&=nC_{n-1}(x)+\sum_{r=1}^{n-1}\binom{n}{r-1}C_{n-r}(x)C_{r-1}(x)\\&=nC_{n-1}(x)+\sum_{k=0}^{n-2}\binom{n}{k}C_{n-k-1}(x)C_k(x)\\&=\sum_{k=0}^{n-1}\binom{n}{k}C_k(x)C_{n-k-1}(x).\end{aligned}$$

Then

$$\sum_{k=0}^{n-1}\binom{n}{k}C_k(x)C_{n-k-1}(x)=(n+(n-1)x)C_{n-1}(x)+x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}C_{n-1}(x).$$

Therefore, by Lemma 4, we immediately get the following result.

Proposition 5. *The binomial convolutions $\sum_{k=0}^{n-1}\binom{n}{k}C_k(x)C_{n-k-1}(x)$ have only real zeros.*

The $n\times n$ *lower Hessenberg matrix* H_n is defined as follows:

$$H_n=\begin{pmatrix}h_{11}&h_{12}&0&\cdots&0&0\\h_{21}&h_{22}&h_{23}&\cdots&0&0\\h_{31}&h_{32}&h_{33}&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\h_{n-1,1}&h_{n-1,2}&h_{n-1,3}&\cdots&h_{n-1,n-1}&h_{n-1,n}\\h_{n,1}&h_{n,2}&h_{n,3}&\cdots&h_{n,n-1}&h_{n,n}\end{pmatrix},$$

where $h_{ij}=0$ if $j>i+1$. Hessenberg matrices frequently appear in numerical analysis [10,32]. Setting $H_0=1$, Cahill et al. [10] gave a recursion for the determinant of the matrix H_n as follows:

$$\det H_n=h_{n,n}\det H_{n-1}+\sum_{r=1}^{n-1}\left((-1)^{n-r}h_{n,r}\prod_{j=r}^{n-1}h_{j,j+1}\det H_{r-1}\right). \tag{7}$$

By induction, comparing (7) and Theorem 3, we discover the following result.

Theorem 6. *For any $n \geqslant 1$, the polynomial $C_n(x)$ can be expressed as the following lower Hessenberg determinant of order n:*

$$C_n(x) = \begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ C_1(x) & \binom{2}{1}x & -1 & \cdots & 0 & 0 \\ C_2(x) & \binom{3}{1}C_1(x) & \binom{3}{2}x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-2}(x) & \binom{n-1}{1}C_{n-3}(x) & \binom{n-1}{2}C_{n-4}(x) & \cdots & \binom{n-1}{n-2}x & -1 \\ C_{n-1}(x) & \binom{n}{1}C_{n-2}(x) & \binom{n}{2}C_{n-3}(x) & \cdots & \binom{n}{n-2}C_1(x) & \binom{n}{n-1}x \end{vmatrix}_{n\times n}.$$

When $n = 4$, from Theorem 6, we see that

$$C_4(x) = \begin{vmatrix} x & -1 & 0 & 0 \\ x & 2x & -1 & 0 \\ x+2x^2 & 3x & 3x & -1 \\ x+8x^2+6x^3 & 4x+8x^2 & 6x & 4x \end{vmatrix} = x + 22x^2 + 58x^3 + 24x^4.$$

3. A class of restricted Stirling permutations and signed permutations

3.1. Context-free grammars and a new differential operator method

A *context-free grammar* (also known as *Chen's grammar* [12,20,38]) G over an alphabet V is defined as a set of substitution rules replacing a letter in V by a formal function over V. The formal derivative D_G with respect to G satisfies the derivation rules:

$$D_G(u+v) = D_G(u) + D_G(v),\ D_G(uv) = D_G(u)v + uD_G(v).$$

So the *Leibniz rule* holds:

$$D_G^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_G^k(u) D_G^{n-k}(v).$$

In [20], Dumont obtained the grammar for Eulerian polynomials by using a grammatical labeling for circular permutations.

Proposition 7 *([20, Section 2.1]). Let $G = \{a \to ab, b \to ab\}$. For any $n \geqslant 1$, one has*

$$D_G^n(a) = D_G^n(b) = a^{n+1} A_n\left(\frac{b}{a}\right).$$

In the following three examples, we introduce a new differential operator method, which will help us to find grammars. We first establish a connection between (1) and Proposition 7.

Example 8. Setting

$$T = x\frac{\mathrm{d}}{\mathrm{d}x},\ a = \frac{1}{1-x} \text{ and } b = \frac{x}{1-x},$$

we obtain $T(a) = T(b) = ab$. By (1), we have $T^n(a) = D_G^n(a)$, where $G = \{a \to ab, b \to ab\}$.

The grammar for the second-order Eulerian polynomials was first discovered by Chen-Fu [13] by using a grammatical labeling for Stirling permutations. Here we give a formal derivation.

Example 9. Setting

$$T = \frac{x}{1-x}\frac{\mathrm{d}}{\mathrm{d}x},\ a = \frac{x}{1-x},\ b = \frac{1}{1-x},$$

we get $T(a) = ab^2$ and $T(b) = ab^2$. Let $G = \{a \to ab^2,\ b \to ab^2\}$. It follows from (2) that

$$D_G^n(a) = T^n(a) = b^{2n+1}C_n\left(\frac{a}{b}\right).$$

Following [29, p. 29], one has

$$\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^n \frac{\mathrm{e}^y}{1-\mathrm{e}^{2y}} = \frac{\mathrm{e}^y B_n(\mathrm{e}^{2y})}{(1-\mathrm{e}^{2y})^{n+1}}, \tag{8}$$

where $B_n(x)$ are the type B Eulerian polynomials. Note that $\frac{\mathrm{d}}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}x}$ and $x = \mathrm{e}^y$ is the solution of $x = \frac{\mathrm{d}x}{\mathrm{d}y}$. It follows from (8) that

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{x}{1-x^2} = \frac{xB_n(x^2)}{(1-x^2)^{n+1}}. \tag{9}$$

We now deduce the grammar for the type B Eulerian polynomials.

Example 10. Setting

$$T = x\frac{\mathrm{d}}{\mathrm{d}x},\ a = \frac{x}{\sqrt{1-x^2}} \text{ and } b = \frac{1}{\sqrt{1-x^2}},$$

we get $T(a) = ab^2,\ T(b) = a^2b$. Let $G = \{a \to ab^2,\ b \to a^2b\}$. It follows from (9) that

$$D_G^n(ab) = T^n(ab) = ab^{2n+1}B_n\left(\frac{a^2}{b^2}\right).$$

In equivalent forms, Dumont [19], Haglund-Visontai [25], Chen-Hao-Yang [16] and Ma-Ma-Yeh [38] all showed that

$$D_G^n(x) = D_G^n(y) = D_G^n(z) = C_n(x, y, z), \tag{10}$$

where $G = \{x \to xyz,\ y \to xyz,\ z \to xyz\}$. By the change of grammar $u = x+y+z,\ v = xy+yz+zx$ and $w = xyz$, it is easy to verify that $D_G(u) = 3w,\ D_G(v) = 2uw,\ D_G(w) = vw$. So we get a grammar $H = \{w \to vw, u \to 3w, v \to 2uw\}$. Recently, Chen-Fu [14] discovered that for any $n \geqslant 1$, one has

$$C_n(x, y, z) = D_G^n(x) = D_H^{n-1}(w) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} u^i v^j w^k,$$

where the coefficient $\gamma_{n,i,j,k}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices. Substituting $u \to x + y + z,\ v \to xy + yz + zx$ and $w \to xyz$, one has

$$C_n(x, y, z) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} (x + y + z)^i (xy + yz + zx)^j (xyz)^k,$$

which has recently been generalized to a seventeen-variable polynomial [41, Theorem 15].

3.2. A class of restricted Stirling permutations

The following fundamental result will be used in our discussion.

Lemma 11. *Let $G = \{x \to xyz,\ y \to xyz,\ z \to xyz\}$. Then we have*

$$D_G^n(yz) = \sum_{\sigma \in \mathcal{Q}_{n+1}^{(1)}} x^{\mathrm{plat}\,(\sigma)} y^{\mathrm{des}\,(\sigma)} z^{\mathrm{asc}\,(\sigma)}, \tag{11}$$

where $\mathcal{Q}_{n+1}^{(1)}$ is the set of Stirling permutations over the multiset $\{1, 2, 2, 3, 3, \ldots, n, n, n+1, n+1\}$, i.e., the element 1 appears only once and the other elements appear two times.

Proof. We first introduce a grammatical labeling for $\sigma \in \mathcal{Q}_{n+1}^{(1)}$ as follows:

(L_1) If σ_i is a plateau, then put a superscript label x right after σ_i.
(L_2) If σ_i is a descent, then put a superscript label y right after σ_i;
(L_3) If σ_i is an ascent, then put a superscript label z right after σ_i;

The weight of σ is given by $w(\sigma) = x^{\mathrm{plat}\,(\sigma)} y^{\mathrm{des}\,(\sigma)} z^{\mathrm{asc}\,(\sigma)}$. When $n = 0$, we have $\mathcal{Q}_1^{(1)} = \{{}^z1^y\}$, which corresponds to yz. When $n = 1$, we have $\mathcal{Q}_2^{(1)} = \{{}^z1^z2^x2^y, {}^z2^x2^y1^y\}$, which corresponds to $D_G(yz) = x(y^2z + yz^2)$. When $n = 2$, the weighted elements in $\mathcal{Q}_3^{(1)}$ can be listed as follows:

$$\begin{gathered}{}^z1^z2^x2^z3^x3^y,\ {}^z1^z2^z3^x3^y2^y,\ {}^z1^z3^x3^y2^x2^y,\ {}^z3^x3^y1^z2^x2^y,\\ {}^z2^x2^y1^z3^x3^y,\ {}^z2^x2^z3^x3^y1^y,\ {}^z2^z3^x3^y2^y1^y,\ {}^z3^x3^y2^x2^y1^y.\end{gathered}$$

The sum of weights of these elements is given as follows:

$$D_G^2(yz) = x^2y^3z + 4x^2y^2z^2 + xy^3z^2 + x^2yz^3 + xy^2z^3.$$

So the result holds for $n = 0, 1, 2$. We proceed by induction. Suppose we get all labeled Stirling permutations in $\mathcal{Q}_{n-1}^{(1)}$, where $n \geqslant 3$. Let σ' be obtained from $\sigma \in \mathcal{Q}_{n-1}^{(1)}$ by inserting the pair nn. Then the changes of labelings are illustrated as follows:

$$\begin{aligned}&\cdots\sigma_i^x\sigma_{i+1}\cdots\mapsto\cdots\sigma_i^z n^x n^y\sigma_{i+1}\cdots;\\&\cdots\sigma_i^y\sigma_{i+1}\cdots\mapsto\cdots\sigma_i^z n^x n^y\sigma_{i+1}\cdots;\\&\cdots\sigma_i^z\sigma_{i+1}\cdots\mapsto\cdots\sigma_i^z n^x n^y\sigma_{i+1}\cdots.\end{aligned}$$

In each case, the insertion of the string nn corresponds to the operator D_G. By induction, the action of the formal derivative D_G on the set of weighted Stirling permutations in $\mathcal{Q}_{n-1}^{(1)}$ gives the set of weighted Stirling permutations in $\mathcal{Q}_n^{(1)}$, and so (11) holds. □

Define

$$E_n(x,y,z) = \sum_{\sigma\in\mathcal{Q}_n^{(1)}} x^{\mathrm{plat}\,(\sigma)}y^{\mathrm{des}\,(\sigma)}z^{\mathrm{asc}\,(\sigma)}.$$

By (11), we see that

$$E_{n+1}(x,y,z) = xyz\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)E_n(x,y,z),\ E_1(x,y,z) = yz. \tag{12}$$

Let $E_n(x,y,z) = \sum_{i\geqslant 1, j\geqslant 1} E_{n,i,j}x^iy^jz^{2n-i-j}$ for $n \geqslant 2$. It follows from (12) that

$$E_{n+1,i,j} = iE_{n,i,j-1} + jE_{n,i-1,j} + (2n-i-j+2)E_{n,i-1,j-1}, \tag{13}$$

with $E_{1,0,1} = 1$ and $E_{1,i,j} = 0$ for all $(i,j) \neq (0,1)$. By (10), we find that

$$C_{n+1}(x,y,z) = D_G^n(xyz) = \sum_{k=0}^n\binom{n}{k}D_G^k(x)D_G^{n-k}(yz) = \sum_{k=0}^n\binom{n}{k}C_k(x,y,z)D_G^{n-k}(yz).$$

So the following result is immediate.

Theorem 12. *We have*

$$C_{n+1}(x,y,z) = \sum_{k=0}^n\binom{n}{k}C_k(x,y,z)E_{n-k+1}(x,y,z).$$

It is routine to check that $\#\mathcal{Q}_{n+1}^{(1)} = E_{n+1}(1,1,1) = (2n)!!$. Let $\pm[n] = [n] \cup \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$, where we denote by $\overline{i}$ the negative element $-i$. The *hyperoctahedral group* $\mathfrak{S}_n^B$ is the group of signed permutations on $\pm[n]$ with the property that $\pi\left(\overline{i}\right) = -\pi(i)$ for all $i \in [n]$. It is well known that $\#\mathfrak{S}_n^B = (2n)!!$ (see [5,41]). Therefore,

$$\#\mathcal{Q}_{n+1}^{(1)} = \#\mathfrak{S}_n^B.$$

A natural idea is therefore to investigate the connections between $\mathcal{Q}_{n+1}^{(1)}$ and $\mathfrak{S}_n^B$.

3.3. Six-variable polynomials over restricted Stirling permutations

Let $w = w_1 w_2 \cdots w_n$ be a word of length n, where w_i are all integers. Except where explicitly stated, we always assume that $w_0 = w_{n+1} = 0$. Let

$$\mathrm{Ap}(w) = \{w_i \mid w_{i-1} < w_i = w_{i+1} \ \&\ 2 \leqslant i \leqslant n-1\},$$
$$\mathrm{Lap}(w) = \{w_i \mid w_{i-1} < w_i = w_{i+1} \ \&\ 1 \leqslant i \leqslant n-1\}$$

be the sets of ascent-plateaux and left ascent-plateaux of w, respectively. Let $\mathrm{ap}\,(w)$ and $\mathrm{lap}\,(w)$ be the numbers of ascent-plateaux and left ascent-plateaux of w, respectively. The *ascent-plateau polynomials* and the *left ascent-plateau polynomials* on Stirling permutations are respectively defined by

$$M_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)},\ W_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)}.$$

From [37, Proposition 1], we see that

$$2^n A_n(x) = \sum_{i=0}^{n} \binom{n}{i} W_i(x) W_{n-i}(x),\ B_n(x) = \sum_{i=0}^{n} \binom{n}{i} M_i(x) W_{n-i}(x),$$

where $B_n(x)$ is the type B Eulerian polynomial over the hyperoctahedral group $\mathfrak{S}_n^B$.

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n^B$. It should be noted that the n letters appearing in the cycle notation of π are the letters $\pi(1), \pi(2), \ldots, \pi(n)$. We say that i is an *excedance* (resp. *anti-excedance*, *fixed point*, *singleton*) of π if $\pi(|\pi(i)|) > \pi(i)$ (resp. $\pi(|\pi(i)|) < \pi(i)$, $\pi(i) = i$, $\pi(i) = \overline{i}$). Let $\mathrm{exc}\,(\pi)$ (resp. $\mathrm{aexc}\,(\pi)$, $\mathrm{fix}\,(\pi)$, $\mathrm{single}\,(\pi)$, $\mathrm{neg}\,(\pi)$, $\mathrm{cyc}\,(\pi)$) be the number of excedances (resp. anti-excedances, fixed points, singletons, negative entries, cycles) of π.

Example 13. The signed permutation $\pi = 68\overline{3}15\overline{7}24\overline{9}$ can be written as $(\overline{9})(\overline{3})(1,6,\overline{7},2,8,4)(5)$. So π has two singletons $\overline{9}$ and $\overline{3}$, one fixed point 5, 3 excedances, 3 anti-excedances, 3 negative entries and 4 cycles.

According to [5, Corollary 3.16], we have

$$B_n(x)=\sum_{\pi\in\mathfrak{S}_n^B}x^{\mathrm{exc}\,(\sigma)+\mathrm{single}\,(\sigma)}=\sum_{\pi\in\mathfrak{S}_n^B}x^{\mathrm{exc}\,(\sigma)+\mathrm{fix}\,(\sigma)}.$$

Lemma 14 *([40, Lemma 5.3]). Let p and q be two given parameters. If*

$$G=\{I\rightarrow qI(t+sp),s\rightarrow(1+p)xy,t\rightarrow(1+p)xy,x\rightarrow(1+p)xy,y\rightarrow(1+p)xy\},\quad(14)$$

then we have

$$D_G^n(I)=I\sum_{\sigma\in\mathfrak{S}_n^B}x^{\mathrm{exc}\,(\sigma)}y^{\mathrm{aexc}\,(\sigma)}s^{\mathrm{single}\,(\sigma)}t^{\mathrm{fix}\,(\sigma)}p^{\mathrm{neg}\,(\sigma)}q^{\mathrm{cyc}\,(\sigma)}.$$

For a word $w=w_1w_2\cdots w_n$, we have the following definitions:

- An *even indexed entry* of w is an element w_i such that the first appearance of w_i (when we read w from left to right) occurs at an even position;
- A *right-to-left minimum* of w is an element w_i such that $w_i\leqslant w_j$ for every $j\in\{i+1,i+2,\ldots,n\}$ and w_i is the first appearance from left to right;
- A *left-to-right minimum* of w is an element w_i such that $w_i<w_j$ for every $j\in\{1,2,\ldots,i-1\}$ or $i=1$;
- A *block* of w is defined as a maximal substring that begins with a left-to-right minimum and contains no other left-to-right minimum.

Let $\mathrm{Even}(w)$ and $\mathrm{Rlmin}(w)$ be the sets of even indexed entries and right-to-left minima of w, respectively. For instance,

$$\mathrm{Even}(4\mathbf{5}54\mathbf{1}\mathbf{2}2\mathbf{3}77\mathbf{3}\mathbf{6}6)=\{2,3,5,6\},\ \mathrm{Rlmin}(4554\mathbf{1}\mathbf{2}2\mathbf{3}77\mathbf{3}\mathbf{6}6)=\{1,2,3,6\}.$$

It is easily derived by induction that w has a unique decomposition as a sequence of blocks. For $\sigma\in\mathcal{Q}_n^{(1)}$, there exists one block beginning with the element 1. For example, the block decomposition of 88346643991255277 is given by [88][34664399][1255277]. We use $\mathrm{even}\,(\sigma)$, $\mathrm{rlmin}(\sigma)$, $\mathrm{lrmin}(\sigma)$ and $\mathrm{bk}_2(\sigma)$ to denote the numbers of even indexed entries, right-to-left minima, left-to-right minima and blocks of size exactly 2 of σ, respectively. When σ_i and σ_{i+1} constitute a block with size exactly 2, it is clear that $\sigma_i=\sigma_{i+1}$. For example, $\mathrm{bk}_2(33221)=2$. The block statistic has been studied by Kuba-Panholzer [33] and Remmel-Wilson [42].

We now introduce three new statistics.

Definition 15. For $\sigma\in\mathcal{Q}_n^{(1)}$, we say that an entry σ_i is

- a *proper ascent-plateau* if σ_i is an ascent plateau, but it is not a right-to-left minimum;

- an *improper ascent-plateau* if $\sigma_i \in \mathrm{Ap}(\sigma) \cap \mathrm{Rlmin}(\sigma)$;
- a *trace* if there exists $2 \leqslant k \leqslant n$, when the subword of σ restricted to $\{1, 2, 2, \ldots, k, k\}$, σ_i is an *improper ascent-plateau* of this subword or σ_i is the second entry that appears in a block of size exactly 2.

Let $\mathrm{Pap}(\sigma), \mathrm{Impap}(\sigma)$ and $\mathrm{Trace}(\sigma)$ be the sets of proper ascent-plateaux, improper ascent-plateaux and traces of σ, respectively. For instance,

$$\mathrm{Pap}(884554122377366) = \{5, 7\},$$
$$\mathrm{Impap}(884554122377366) = \{2, 6\},$$
$$\mathrm{Trace}(884554122377366) = \{2, 3, 4, 6, 8\}.$$

It is clear that

$$\mathrm{AP}(\sigma) = \mathrm{Pap}(\sigma) \cup \mathrm{Impap}(\sigma),\ \mathrm{Pap}(\sigma) \cap \mathrm{Impap}(\sigma) = \emptyset.$$

We use $\mathrm{pap}(\sigma)$ and $\mathrm{impap}(\sigma)$ to denote the numbers of proper and improper ascent-plateaux of σ, respectively. Hence $\mathrm{ap}\,(\sigma) = \mathrm{pap}(\sigma) + \mathrm{impap}(\sigma)$. The *complement index of ascent-plateaux* of σ is defined by

$$\mathrm{cap}(\sigma) = n - \mathrm{pap}(\sigma) - \mathrm{impap}(\sigma) - \mathrm{bk}_2(\sigma) = n - \mathrm{ap}\,(\sigma) - \mathrm{bk}_2(\sigma).$$

We can now conclude the main result of this section.

Theorem 16. *We have*

$$\sum_{\sigma \in \mathcal{Q}_{n+1}^{(1)}} x^{\mathrm{pap}(\sigma)} y^{\mathrm{cap}(\sigma)} s^{\mathrm{impap}(\sigma)} t^{\mathrm{bk}_2(\sigma)} p^{\mathrm{even}\,(\pi)} q^{\mathrm{tr}(\sigma)}$$
$$= \sum_{\pi \in \mathfrak{S}_n^B} x^{\mathrm{exc}\,(\pi)} y^{\mathrm{aexc}\,(\pi)} s^{\mathrm{single}\,(\pi)} t^{\mathrm{fix}\,(\pi)} p^{\mathrm{neg}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}.$$

Proof. Recall that $\mathcal{Q}_n^{(1)}$ is the set of Stirling permutations over the multiset $\{1, 2, 2, 3, 3, \ldots, n, n\}$. A grammatical labeling for $\sigma \in \mathcal{Q}_n^{(1)}$ is given as follows:

(L_1) we use the superscript I to mark the first position (just before σ_1) and the last position (at the end of σ), and denoted by $\overbrace{\sigma_1\sigma_2\cdots\sigma_{2n-1}}^{I}$;

(L_2) if σ_i is a proper ascent-plateau, then we label the two positions just before and right after σ_i by a subscript label x;

(L_3) if σ_i is an improper ascent-plateau, then we label the two positions just before and right after σ_i by a subscript label s;

(L_4) if σ_i and σ_{i+1} constitute a block with size exactly 2, then we label the two positions just before and right after σ_{i+1} by a subscript label t;
(L_5) except the above labeled positions, there are still even number of positions, and we use a y to label pairwise nearest elements from left to right;
(L_6) we attach a superscript label p to every even indexed entry;
(L_7) we attach a subscript label q to each trace.

With this labeling, the weight of σ is defined as the product of the labels, i.e.,

$$w(\sigma)=x^{\mathrm{pap}(\sigma)}y^{\mathrm{cap}(\sigma)}s^{\mathrm{impap}(\sigma)}t^{\mathrm{bk}_2(\sigma)}p^{\mathrm{even}\,(\pi)}q^{\mathrm{tr}(\sigma)}.$$

For example, the grammatical labeling for 884554122377366 is given as follows:

$$\overbrace{8\underbrace{\;8_q\;}_{t}4\underbrace{\;5^p\;}_{x}5\overbrace{\;4_q\;}^{y}1\underbrace{\;2^p_q\;}_{s}2\overbrace{3^p_q\underbrace{\;7\;}_{x}7}^{y}3\underbrace{\;6^p_q\;}_{s}6}^{I}.$$

We proceed by induction. The element in $\mathcal{Q}_1^{(1)}$ can be labeled as $\overbrace{\;1\;}^{I}$. The labeled elements in $\mathcal{Q}_2^{(1)}$ can be listed as follows:

$$\overbrace{1\underbrace{\;2^p_q\;}_{s}2}^{I},\ \overbrace{2\underbrace{\;2_q\;}_{t}1}^{I}.$$

Let G be given by (14). Note that $D_G(I)=qI(t+sp)$. Hence the result holds for $n=1$. Suppose we get all labeled Stirling permutations in $\sigma\in\mathcal{Q}_{n-1}^{(1)}$, where $n\geqslant 2$. Let $\widehat{\sigma}$ be obtained from σ by inserting the string nn. There are six possibilities to label the inserted string and to relabel some elements of σ:

(c_1) by the definition of trace, we never need to relabel the subscript label q.
(c_2) if nn is inserted immediately before or right after σ, then the changes of labelings are illustrated as follows:

$$\overbrace{\;\sigma\;}^{I}\to\overbrace{n\underbrace{\;n_q\;}_{t}\sigma}^{I},\quad \overbrace{\;\sigma\;}^{I}\to\overbrace{\sigma\underbrace{\;n^p_q\;}_{s}n}^{I}.$$

(c_3) if nn is inserted immediately before or right after an element with a label t. Since ii forms a block of size 2 which means the entries before ii are all bigger than i, thus the first i located at an odd position. The changes of labelings are illustrated as follows:

$$\cdots i \underbrace{\ i_q\ }_{t} \cdots \to \cdots i \underbrace{\ n^p\ }_{x} n \underbrace{\ i_q\ }_{y} \cdots, \quad \cdots i \underbrace{\ i_q\ }_{t} \cdots \to \cdots\cdots i \overbrace{i_q \underbrace{\ n\ }_{x} n}^{y} \cdots\cdots .$$

(c_4) if nn is inserted immediately before or right after an element with label s, then the changes of labelings are illustrated as follows:

$$\cdots \underbrace{\ i_q^p\ }_{s} i \cdots \to \cdots \underbrace{\ n^p\ }_{x} n \overbrace{\ i_q^p\ }^{y} i \cdots, \quad \cdots \underbrace{\ i_q^p\ }_{s} i \cdots \to \cdots \overbrace{i_q^p \underbrace{\ n\ }_{x} n}^{y} i \cdots .$$

(c_5) if nn is inserted immediately before or right after an element with label x, then the changes of labelings are illustrated as follows:

$$\cdots \underbrace{\ i^p\ }_{x} i \cdots \to \cdots \underbrace{\ n^p\ }_{x} n \overbrace{\ i^p\ }^{y} i \cdots, \quad \cdots \underbrace{\ i^p\ }_{x} i \cdots \to \cdots \overbrace{i^p \underbrace{\ n\ }_{x} n}^{y} i \cdots ;$$

$$\cdots \underbrace{\ i\ }_{x} i \cdots \to \cdots \underbrace{\ n\ }_{x} n \overbrace{\ i\ }^{y} i \cdots, \quad \cdots \underbrace{\ i\ }_{x} i \cdots \to \cdots \overbrace{i \underbrace{\ n^p\ }_{x} n}^{y} i \cdots .$$

(c_6) if nn is inserted into either position of a pair labeled by y, then the first n always get a label x and there still two positions get a label y. For each pair of positions labeled by y, there is an odd number of entries between them. So we get the substitution rule $y \to (1+p)xy$.

In each case, the insertion of nn corresponds to one substitution rule in G. Therefore, the action of D_G on the set of weighted Stirling permutations in $\mathcal{Q}_{n-1}^{(1)}$ gives the set of weighted Stirling permutations in $\mathcal{Q}_n^{(1)}$. So we complete the proof by Lemma 14. $\square$

We now collect several well-studied Eulerian-type polynomials. Let

$$B_n(x,p,q) = \sum_{\pi \in \mathfrak{S}_n^B} x^{\mathrm{exc}\,(\pi)+\mathrm{single}\,(\pi)} p^{\mathrm{neg}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}$$

be a (p,q)-Eulerian polynomial of type B. The *types A and B derangement polynomials* are respectively defined by

$$d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\mathrm{exc}\,(\pi)},\ d_n^B(x) = \sum_{\pi \in \mathcal{D}_n^B} x^{\mathrm{exc}\,(\pi)},$$

where $\mathcal{D}_n = \{\pi \in \mathfrak{S}_n \mid \mathrm{fix}\,(\pi) = 0\}$ and $\mathcal{D}_n^B = \{\pi \in \mathfrak{S}_n^B \mid \mathrm{fix}\,(\pi) = 0\}$. These polynomials have been extensively studied. See [6,17,27,40] for recent progress.

Corollary 17. *We have*

$$B_n(x,p,q)=\sum_{\sigma\in\mathcal{Q}_{n+1}^{(1)}} x^{\mathrm{ap}(\sigma)}p^{\mathrm{even}\,(\pi)}q^{\mathrm{tr}(\sigma)},$$

$$d_n(x)=\sum_{\substack{\pi\in\mathcal{Q}_{n+1}^{(1)}\\ \mathrm{bk}_2(\pi)=0\\ \mathrm{even}\,(\pi)=0}} x^{\mathrm{pap}(\pi)},\ d_n^B(x)=\sum_{\substack{\pi\in\mathcal{Q}_{n+1}^{(1)}\\ \mathrm{bk}_2(\pi)=0}} x^{\mathrm{pap}(\pi)}.$$

The (p,q)-*Eulerian polynomials* $A_n(x,p,q)$ are defined by

$$A_n(x,p,q)=\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)}p^{\mathrm{fix}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}.$$

Combining [40, Theorem 5.2] and Theorem 16, we get the following result.

Corollary 18. *We have*

$$\sum_{\sigma\in\mathcal{Q}_{n+1}^{(1)}} x^{\mathrm{pap}(\sigma)}y^{\mathrm{cap}(\sigma)}s^{\mathrm{impap}(\sigma)}t^{\mathrm{bk}_2(\sigma)}p^{\mathrm{even}\,(\pi)}q^{\mathrm{tr}(\sigma)}=(1+p)^n y^n A_n\left(\frac{x}{y},\frac{t+sp}{y+py},q\right).$$

In particular, we have

$$\sum_{\sigma\in\mathcal{Q}_{n+1}^{(1)}} s^{\mathrm{impap}(\sigma)}t^{\mathrm{bk}_2(\sigma)}=\sum_{\pi\in\mathfrak{S}_n}(t+s)^{\mathrm{fix}\,(\pi)}2^{n-\mathrm{fix}\,(\pi)}.$$

4. Box sorting algorithm and standard Young tableaux

4.1. Preliminaries

The *Weyl algebra* W is the unital algebra generated by two symbols D and U satisfying the commutation relation $DU-UD=I$, where I is the identity which we identify with "1". Any word w in the letters U,D can always be brought into *normal ordered form* where all letters D stand to the right of all the letters U. A famous example of W is given by the substitutions: $D\to\frac{\mathrm{d}}{\mathrm{d}x}$, $U\to x$. Except as otherwise indicated, we always let $D=\frac{\mathrm{d}}{\mathrm{d}x}$. As early as 1823, Scherk [3, Appendix A] found that

$$(xD)^n=\sum_{k=0}^{n}\left\{{n\atop k}\right\}x^kD^k. \tag{15}$$

According to [3, Proposition A.2], one has

$$(\mathrm{e}^xD)^n=\mathrm{e}^{nx}\sum_{k=0}^{n}\left[{n\atop k}\right]D^k,$$

where $\left[{n \atop k}\right]$ is the (signless) Stirling number of the first kind, i.e., the number of permutations in $\mathfrak{S}_n$ with k cycles. Many generalizations of (15) occur naturally in quantum physics, combinatorics and algebra. The reader is referred to [43] for a survey of this topic.

Throughout this paper, we always let $c := c(x)$ and $f := f(x)$ be two smooth functions in the indeterminate x. We adopt the convention that $c_k = D^k c$ and $\mathbf{f}_k = D^k f$ for $k \geqslant 0$. Set $\mathbf{f}_0 = f$ and $c_0 = c$, where $D = \frac{\mathrm{d}}{\mathrm{d}x}$. The first few $(cD)^n f$ are given as follows:

$$\begin{aligned}
&(cD)f = (c)\mathbf{f}_1,\ (cD)^2 f = (cc_1)\mathbf{f}_1 + (c^2)\mathbf{f}_2,\\
&(cD)^3 f = (cc_1^2 + c^2c_2)\mathbf{f}_1 + (3c^2c_1)\mathbf{f}_2 + (c^3)\mathbf{f}_3,\\
&(cD)^4 f = (cc_1^3 + 4c^2c_1c_2 + c^3c_3)\mathbf{f}_1 + (7c^2c_1^2 + 4c^3c_2)\mathbf{f}_2 + (6c^3c_1)\mathbf{f}_3 + (c^4)\mathbf{f}_4.
\end{aligned}$$

For $n \geqslant 1$, we define

$$(cD)^n f = \sum_{k=1}^{n} F_{n,k}\mathbf{f}_k. \tag{16}$$

Note that $F_{n,k} = F_{n,k}(c, c_1, \ldots, c_{n-k})$ is a function with variables $c, c_1, \ldots, c_{n-k}$. In particular, $F_{1,1} = c$, $F_{2,1} = cc_1$ and $F_{2,2} = c^2$. Clearly, $F_{n+1,1} = cDF_{n,1}$, $F_{n,n} = c^n$ and for $2 \leqslant k \leqslant n$, we have the recurrence relation

$$F_{n+1,k} = cF_{n,k-1} + cDF_{n,k}.$$

By induction, Comtet [18] found an explicit formula of $F_{n,k}$. Recently, Briand-Lopes-Rosas [7] gave a survey of the combinatorial properties of $F_{n,k}$, which can be summarized as follows.

Proposition 19 *([7]). Let $F_{n,k}$ be defined by (16). There exist positive integers $a(n,\lambda)$ such that*

$$F_{n,k} = \sum_{\lambda \vdash n-k} a(n,\lambda)c^{n-\ell(\lambda)}c_\lambda,$$

where λ runs over all partitions of $n-k$. The Stirling numbers of the first and second kinds, and the Eulerian numbers can be respectively expressed as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\lambda \vdash n-k} a(n,\lambda),\ \begin{Bmatrix} n \\ k \end{Bmatrix} = a(n, 1^{n-k}),\ \left\langle {n \atop k} \right\rangle = \sum_{\ell(\lambda)=n-k} a(n,\lambda).$$

In [26], Han-Ma first gave a simple proof of Comtet's formula via inversion sequences, and then introduced k-Young tableaux and their g-indices. Using the indispensable k-Young tableaux, Han-Ma obtained a unified combinatorial interpretation of Eulerian

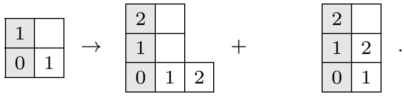


Fig. 1. An illustration of the change of weights: $cc_1 \to c^2c_2 + cc_1^2$.

polynomials and second-order Eulerian polynomials. In practice, however, it seems to be difficult to understand the definition of g-index (see [26, Definition 2.5, Lemma 3.1]). In order to clarify the nature of g-index, we shall introduce box sorting algorithm in the next subsection.

4.2. Box sorting algorithm and the nature of g-index

Rota [31] once said "I will tell you shamelessly what my bottom line is: It is placing balls into boxes". As discussed before, we always let $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and $c = c(x)$. In order to study the powers of cD, we shall introduce the box sorting algorithm.

An *ordered weak set partition* of $[n]$ is a list of pairwise disjoint subsets (maybe empty) of $[n]$ such that the union of these subsets is $[n]$. These subsets are called the *parts* of the partition. A *weak composition* α of an integer n, denoted by $\alpha \models n$, with m parts is a way of writing n as the sum of any sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of nonnegative integers. For $\alpha \models n$, the *Young weak composition diagram* of α, also denoted by α, is the left-justified array of n boxes with α_i boxes in the i-th row. We follow the French convention, which means that we number the rows from bottom to top, and the columns from left to right. The box in the i-th row and j-th column is denoted by the pair (i, j). A *Young weak composition tableau* (YWCT, for short) of shape α is obtained by placing the integers $\{1, 2, \ldots, n\}$ into n boxes of the diagram such that each of those integers is used exactly once. We will often identify an ordered weak set partition with the corresponding YWCT. It should be noted that there may be some empty boxes in YWCT. In the following discussion, we always put a special column of $n+1$ boxes at the left of YWCT or SYT, and labeled by $0, 1, 2, \ldots, n$ from bottom to top. See Fig. 1 and Table 1 for instance.

The following label schema is fundamental.

Label schema. Let p be an ordered weak set partition of $[n]$. We give a labeling of p as follows. Label the i-th subset by the subscript $c_{(i-1)}$, and label a subset with i elements by a superscript c_i, where $i \geqslant 1$. Moreover, if the i-th subset is empty, we always label it by a superscript c. The *weight* of p is defined as the product of the superscript labels.

We can rewrite $(cD)^n c$ as follows:

$$\left(c_{(n)}D_{(n)}\right)\left(c_{(n-1)}D_{(n-1)}\right)\cdots\left(c_{(2)}D_{(2)}\right)\left(c_{(1)}D_{(1)}\right)c_{(0)}, \tag{17}$$

where $c_{(0)} = c_{(i)} = c$ and $D_{(i)} = D$ for all $i \in [n]$. A crucial observation is that the differential operator $D_{(i)}$ in (17) can only applied to $c_{(k)}$, where $0 \leqslant k \leqslant i-1$.

Table 1
$\phi^{-1}(T) = \{p \in \mathrm{OWP}_3 \,|\, \phi(p) = T\}$, where $T \in \mathrm{SYT}\,(3)$.

T	$\overset{\phi^{-1}}{\Longrightarrow}$		p	count
$\begin{array}{llll} 3 & & & \\ 2 & & & \\ 1 & & & \\ 0 & 1 & 2 & 3 \end{array}$	$\Longrightarrow$	$\begin{array}{llll} 3 & \square & & \\ 2 & \square & & \\ 1 & \square & & \\ 0 & 1 & 2 & 3 \end{array}$	$(\{1,2,3\},\{\},\{\},\{\})$	1
$\begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 2 & \\ 0 & 1 & 3 \end{array}$	$\Longrightarrow$	$\begin{array}{lll} 3 & \square & \\ 2 & \square & \\ 1 & 2 & \\ 0 & 1 & 3 \end{array} + \begin{array}{lll} 3 & \square & \\ 2 & \square & \\ 1 & 2 & 3 \\ 0 & 1 & \end{array}$	$(\{1,3\},\{2\},\{\},\{\}),\ (\{1\},\{2,3\},\{\},\{\})$	2
$\begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 3 & \\ 0 & 1 & 2 \end{array}$	$\Longrightarrow$	$\begin{array}{lll} 3 & \square & \\ 2 & \square & \\ 1 & 3 & \\ 0 & 1 & 2 \end{array} + \begin{array}{lll} 3 & \square & \\ 2 & 3 & \\ 1 & \square & \\ 0 & 1 & 2 \end{array}$	$(\{1,2\},\{3\},\{\},\{\}),\ (\{1,2\},\{\},\{3\},\{\})$	2
$\begin{array}{ll} 3 & \\ 2 & 3 \\ 1 & 2 \\ 0 & 1 \end{array}$	$\Longrightarrow$	$\begin{array}{ll} 3 & \square \\ 2 & 3 \\ 1 & 2 \\ 0 & 1 \end{array}$	$(\{1\},\{2\},\{3\},\{\})$	1

When $n = 1$, we have $(cD)c = \left(c_{(1)}D_{(1)}\right) c_{(0)} = cc_1$. When $n = 2$, we have $(cD)^2c = \left(c_{(2)}D_{(2)}\right) \left(c_{(1)}D_{(1)}\right) c_{(0)} = cc_1^2 + c^2c_2$.

Next, we introduce the box sorting algorithm, designed to transform a term in the expansion of (17) into an ordered weak set partition, which can also be represented by a YWCT. When multiply a new term $c_{(i)}D_{(i)}$, the procedure can be summarized as follows:

- When applying $D_{(i)}$ to $c_{(j)}$, it corresponds to the insertion of the element i into the box with the subscript $c_{(j)}$;
- Multiplying by $c_{(i)}$ corresponds to the opening of a new empty box $\{\}^{c}_{c_{(i)}}$.

We now provide a detailed description of the *box sorting algorithm*. Start with an empty box $(\{\}^{c}_{c_{(0)}})$. We proceed as follows:

$BS1$: When $n = 1$, we first insert the element 1 to the empty box, which corresponds to the operation $D_{(1)}(c_{(0)})$. We then open a new empty box, which corresponds to the multiplication by $c_{(1)}$. Thus we get $(\{1\}^{c_1}_{c_{(0)}}, \{\}^{c}_{c_{(1)}})$.

$BS2$: When $n = 2$, we distinguish two cases: (i) we first insert the element 2 into the first box $\{1\}^{c_1}_{c_{(0)}}$, which corresponds to apply the operation $D_{(2)}$ to $c_{(0)}$. We then open a new empty box, which corresponds to the multiplication by $c_{(2)}$; (ii) We first insert the element 2 into the empty box $\{\}^{c}_{c_{(1)}}$, which corresponds to apply the operation $D_{(2)}$ to $c_{(1)}$. We then open a new empty box, which corresponds to

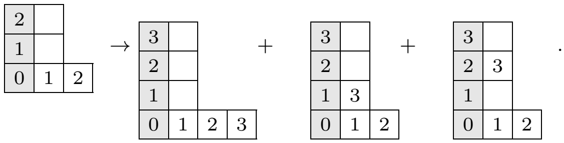


Fig. 2. The insertions of 3 into $(\{1,2\}^{c_2}_{c_{(0)}}, \{\}^{c}_{c_{(1)}}, \{\}^{c}_{c_{(2)}})$.

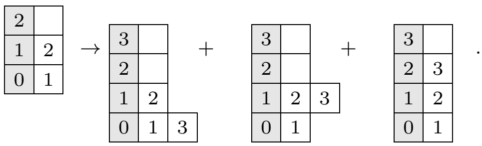


Fig. 3. The insertions of 3 into $(\{1\}^{c_1}_{c_{(0)}}, \{2\}^{c_1}_{c_{(1)}}, \{\}^{c}_{c_{(2)}})$.

the multiplication by $c_{(2)}$. Therefore, we get the following correspondence between ordered weak set partitions and their weights:

$$c^2c_2 \leftrightarrow (\{1,2\}^{c_2}_{c_{(0)}}, \{\}^{c}_{c_{(1)}}, \{\}^{c}_{c_{(2)}}), \quad cc_1^2 \leftrightarrow (\{1\}^{c_1}_{c_{(0)}}, \{2\}^{c_1}_{c_{(1)}}, \{\}^{c}_{c_{(2)}}).$$

The process from $BS1$ to $BS2$ can be illustrated by Fig. 1.

$BS3$: If all of the elements $[i-1]$ have already been inserted, then we consider the insertion of i, where $i \geqslant 3$. Suppose that we insert the element i into the k-th box, which has the label $\{\}^{c_\ell}_{c_{(k-1)}}$, where $1 \leqslant k \leqslant i$. Then this insertion corresponds to apply $D_{(i)}$ to $c_{(k-1)}$, and the label of the k-th box becomes $\{\}^{c_{\ell+1}}_{c_{(k-1)}}$. We then open a new empty box, which corresponds to the multiplication by $c_{(i)}$. When $i=3$, see Figs. 2 and 3 for illustrations, where each empty box in the first column of a YWCT corresponds to an empty subset.

Definition 20. Let OWP_n denote the collection of ordered weak set partitions of $[n]$ into $n+1$ blocks $B_0 \cup B_1 \cup \cdots \cup B_n$ for which the following conditions hold: (a) $1 \in B_0$; (b) if B_i is nonempty, then its minimum is larger than i, where $1 \leqslant i \leqslant n$.

Suppose $p \in \mathrm{OWP}_n$. It is clear that the $(n+1)$-th block B_n must be empty. Denote by $w_i(p)$ the number of blocks in p with i elements. The *weight function* of p is defined by

$$w(p) = \prod_{i=0}^{n} c_i^{w_i(p)}. \tag{18}$$

By the box sorting algorithm, we immediately get the following result.

Lemma 21. *For $n \geqslant 1$, we have $(cD)^n c = \sum_{p \in \mathrm{OWP}_n} w(p)$.*

Suppose $T \in \mathrm{SYT}\,(n)$. We define $w_i(T)$ to be the number of rows in T with i elements. Let $\ell(\lambda(T))$ be the number of rows of T, where λ is the shape of T. Then $\ell(\lambda(T)) = \sum_{i=1}^{n} w_i(T)$ and $n = \sum_{i=1}^{n} iw_i(T)$. The *weight function* of T is defined by

$$w(T) = c^{n+1-\ell(\lambda(T))} \prod_{i=1}^{n} c_i^{w_i(T)}. \tag{19}$$

Let ϕ be the map from OWP_n to $\mathrm{SYT}\,(n)$, which is described as follows:

$OS1$: For $p \in \mathrm{OWP}_n$, let Y be the corresponding YWCT . Reorder the left-justified rows of Y by their length in decreasing order from the bottom to the top, and delete all empty boxes.

$OS2$: Rearrange the entries in each column in ascending order from the bottom to the top.

In view of (18) and (19), we see that for any $p \in \mathrm{OWP}_n$, one has $\phi(p) \in \mathrm{SYT}\,(n)$ and

$$w(p) = w\left(\phi(p)\right). \tag{20}$$

Definition 22. Suppose $T \in \mathrm{SYT}\,(n)$. Let $\phi^{-1}(T) = \{p \in \mathrm{OWP}_n \mid \phi(p) = T\}$. We call $\#\phi^{-1}(T)$ the *g-index* of T.

Clearly, $\#\phi^{-1}(T) = 1$ for $T \in \mathrm{SYT}\,(1)$ or $T \in \mathrm{SYT}\,(2)$. The correspondence between $\mathrm{SYT}\,(3)$ and OWP_3 is listed in Table 1. For $T \in \mathrm{SYT}\,(n)$, let T_i be the element in $\mathrm{SYT}\,(i)$ obtained from T by deleting the $n-i$ elements $i+1, i+2, \ldots, n$. We denote by $\mathrm{col}_k(T_i)$ the size of the k-th column of T_i.

Theorem 23. *For $T \in \mathrm{SYT}\,(n)$, the g-index of T can be computed as follows:*

$$\#\phi^{-1}(T) = \prod_{i=1}^{n} \sigma_i(T), \tag{21}$$

where $\sigma_i(T)$ is defined by

$$\sigma_i(T) = \begin{cases} i - \mathrm{col}_1(T_i) + 1, & \text{if } i \text{ is in the first column of } T; \\ \mathrm{col}_k(T_i) - \mathrm{col}_{k+1}(T_i) + 1, & \text{if } i \text{ is in the } (k+1)\text{-th column of } T, \text{ where } k \geqslant 1. \end{cases}$$

We call $\sigma_i(T)$ the g-index of the element i. Then we have

$$(cD)^n c = \sum_{T \in \mathrm{SYT}\,(n)} \#\phi^{-1}(T) w(T) = \sum_{T \in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T) c_i^{w_i(T)} \right) c^{n+1-\ell(\lambda(T))}. \tag{22}$$

Proof. In order to prove (21), we need to count the possible positions of each entry i of $p \in \phi^{-1}(T)$. We distinguish two cases:

(i) Suppose that i is the r-th entry in the first column of T. Then $r = \mathrm{col}_1(T_i)$. In T_i, the entry i is the maximum. By the box sorting algorithm, we see that there are $i-(r-1)$ ways to insert i, and each insertion generates an element of OWP_i. See Table 1 for an illustration.

(ii) Suppose that i is the r-th entry in the $(k+1)$-th column of T, where $k \geqslant 1$. Then $r = \mathrm{col}_{k+1}(T_i)$. In T_i, the entry i is the maximum. By the box sorting algorithm, we find that there are $\mathrm{col}_k(T_i)-(r-1)$ ways to insert the entry i, and each insertion generates an element of OWP_i.

Continuing in this way, we eventually recover all the elements in OWP_n. By the multiplication principle, we get (21). Combining (20), (21) and Lemma 21, we arrive at (22), and hence the proof is complete. □

Remark 24. The definition of g-index was first introduced in [26, p. 1443]. It should be noted that (22) only implicitly follows from [26, Lemma 3.1], which was obtained by using a relationship between k-Young tableaux and standard Young tableaux. In this paper, with the aid of box sorting algorithm, we give an adequate explanation for the definition of g-index. Moreover, in Definition 30, we introduce the second-order g-index.

By an elusive relationship between k-Young tableaux and standard Young tableaux, the following two results have been obtained in [26]. We give a direct proof of them for our purpose.

Corollary 25 *([26, Theorem 2.11]). We have*

$$A_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T) \right) x^{\ell(\lambda(T))}.$$

Proof. Let $G = \{x \to x, y \to x\}$. Note that Proposition 7 can also be restated as

$$(yD_G)^n(y)|_{y=1} = A_n(x).$$

Taking $c = y$ in (22), we get $c_i = D_G^i(c) = D_G^i(y) = x$ for $i \geqslant 1$. It follows from (22) that

$$\begin{aligned} A_n(x) &= \sum_{T \in \mathrm{SYT}(n)} \left(\prod_{i=1}^{n} \sigma_i(T) c_i^{w_i(T)} \right) c^{n+1-\ell(\lambda(T))} \Bigg|_{c=y=1,\ c_i=x} \\ &= \sum_{T \in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T) \right) x^{\sum_{i=1}^{n} w_i(T)} \end{aligned}$$

$$= \sum_{T\in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T)\right) x^{\ell(\lambda(T))}. \quad \square$$

Corollary 26 *([26, Theorem 2.10]). We have*

$$C_n(x) = \sum_{T\in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T) i!^{w_i(T)}\right) x^{n+1-\ell(\lambda(T))}.$$

Proof. Let $G = \{x \to y^2,\ y \to y^2\}$. From Example 9, one can easily verify that

$$(xD_G)^n(x) = y^{2n+1} C_n\left(\frac{x}{y}\right). \tag{23}$$

Taking $c = x$, then $c_i = D_G^i(c) = D_G^i(x) = i!y^{i+1}$ for $i \geqslant 1$. By (22), we get

$$C_n(x) = \sum_{T\in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T) c_i^{w_i(T)}\right) c^{n+1-\ell(\lambda(T))}\Big|_{c=x,\ c_i=i!y^{i+1},\ y=1},$$

which yields the desired result. This completes the proof. $\square$

In the next subsection, we first give several new applications of Theorem 23, and then we provide a variant of it.

4.3. Further investigations involving Theorem 23

The type B Eulerian polynomials $B_n(x)$ satisfy the recurrence relation

$$B_n(x) = (2nx + 1 - x)B_{n-1}(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_{n-1}(x),$$

with $B_0(x) = 1$ (see [5, Theorem 3.4]). Here we give an expression of $B_n(x)$.

Theorem 27. *Let $c_{2i-1} = 4^{i-1}(1+x)$ and $c_{2i} = 4^i\sqrt{x}$ for $i \geqslant 1$. We have*

$$B_n(x) = \sum_{T\in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T) c_i^{w_i(T)}\right) x^{\frac{1}{2}(n-\ell(\lambda(T)))}.$$

Proof. From Example 10, we see that $xB_n(x^2) = (xyD_G)^n(xy)|_{y=1}$, where $G = \{x \to y,\ y \to x\}$. Taking $c = xy$, we notice that

$$c_{2i-1} = D_G^{2i-1}(c) = 4^{i-1}(x^2+y^2),\ c_{2i} = D_G^{2i}(c) = 4^i xy \text{ for } i \geqslant 1.$$

By (22), we find that

Table 2
The computation of $C_3(x) = x + 8x^2 + 6x^3$.

T	$\sigma_i(T),\ w_i(T)$	enumerator	
$\begin{array}{llll} 3 & & & \\ 2 & & & \\ 1 & & & \\ 0 & 1 & 2 & 3 \end{array}$	$\Longrightarrow$	$\begin{array}{c}\sigma_1(T)=\sigma_2(T)=\sigma_3(T)=1\\ w_1(T)=w_2(T)=0,\ w_3(T)=1\end{array}$	$x + 4x^2 + x^3$
$\begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 2 & \\ 0 & 1 & 3 \end{array}$	$\Longrightarrow$	$\begin{array}{c}\sigma_1(T)=\sigma_2(T)=1,\ \sigma_3(T)=2\\ w_1(T)=w_2(T)=1,\ w_3(T)=0\end{array}$	$2x(x + x^2)$
$\begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 3 & \\ 0 & 1 & 2 \end{array}$	$\Longrightarrow$	$\begin{array}{c}\sigma_1(T)=\sigma_2(T)=1,\ \sigma_3(T)=2\\ w_1(T)=w_2(T)=1,\ w_3(T)=0\end{array}$	$2x(x + x^2)$
$\begin{array}{ll} 3 & \\ 2 & 3 \\ 1 & 2 \\ 0 & 1 \end{array}$	$\Longrightarrow$	$\begin{array}{c}\sigma_1(T)=\sigma_2(T)=\sigma_3(T)=1\\ w_1(T)=3,\ w_2(T)=w_3(T)=0\end{array}$	x^3

$$\begin{aligned} xB_n(x^2) &= \sum_{T\in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T) c_i^{w_i(T)}\right) c^{n+1-\ell(\lambda(T))}\Bigg|_{\substack{c_{2i-1}=4^{i-1}(x^2+y^2),\ c_{2i}=4^i xy,\\ c=xy,\ y=1}} \\ &= \sum_{T\in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T) c_i^{w_i(T)}\right) x^{n+1-\ell(\lambda(T))}\Bigg|_{c_{2i-1}=4^{i-1}(1+x^2),\ c_{2i}=4^i x}, \end{aligned}$$

which yields the desired result. □

We can now give an affirmative answer to Problem 1.

Theorem 28. *We have*

$$C_n(x) = \sum_{T\in \mathrm{SYT}\,(n)} \prod_{i=1}^{n} \sigma_i(T) A_i(x)^{w_i(T)}.$$

Proof. From Example 9, we see that if $G = \{x \to xy,\ y \to xy\}$, then

$$(yD_G)^n(y) = y^{2n+1} C_n\left(\frac{x}{y}\right). \tag{24}$$

Taking $c = y$, by Proposition 7, we see that $c_i|_{y=1} = D_G^i(y)|_{y=1} = A_i(x)$ for $i \geqslant 1$. By (22), we find the desired formula. See Table 2 for an illustration. □

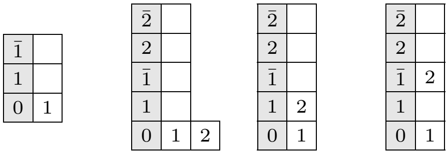


Fig. 4. The illustrations of $(c^2D)\ c = c^2c_1$ and $(c^2D)^2\ c = c^4c_2 + 2c^3c_1^2$ by using YWCT.

Let $\mathrm{SYT}\,(n;k)$ be the subset of $\mathrm{SYT}\,(n)$ with at most k columns.

Theorem 29. *For the trivariate second-order Eulerian polynomials, we have*

$$C_{n+1}(x,y,z) = \sum_{T\in \mathrm{SYT}\,(n;3)} \left(\prod_{i=1}^{n} \sigma_i(T)\right) {c_1}^{w_1(T)}{c_2}^{w_2(T)}6^{w_3(T)}(xyz)^{n+1-\ell(\lambda(T))},$$

where $c_1 = xy + yz + xz$ and $c_2 = 2x + 2y + 2z$. In particular, setting $y = z = 1$, we obtain that

$$C_{n+1}(x) = \sum_{T\in \mathrm{SYT}\,(n;3)} \left(\prod_{i=1}^{n} \sigma_i(T)\right) (1+2x)^{w_1(T)}(4+2x)^{w_2(T)}6^{w_3(T)}x^{n+1-\ell(\lambda(T))}.$$

Proof. It follows from (10) that $(xyzD_G)^n(xyz) = C_{n+1}(x,y,z)$, where

$$G = \{x \to 1,\ y \to 1, z \to 1\}.$$

Setting $c = xyz$, we get that $c_1 = xy+yz+xz$, $c_2 = 2x+2y+2z$, $c_3 = D_G(2x+2y+2z) = 6$, and $c_i = 0$ for $i \geqslant 4$. Substituting $c = xyz,\ c_1 = xy + yz + xz,\ c_2 = 2x + 2y + 2z$, $c_3 = 6$, and $c_i = 0$ for $i \geqslant 4$ into the following expression:

$$C_{n+1}(x,y,z) = \sum_{T\in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \sigma_i(T)c_i^{w_i(T)}\right) c^{n+1-\ell(\lambda(T))},$$

we obtain the desired formula. This completes the proof. □

In the sequel, we shall provide a variant of Theorem 23. Note that $(c^2D)^nc$ can be rewritten as follows:

$$\left(c_{(2n)}c_{(2n-1)}D_{(n)}\right)\left(c_{(2n-2)}c_{(2n-3)}D_{(n-1)}\right)\cdots\left(c_{(4)}c_{(3)}D_{(2)}\right)\left(c_{(2)}c_{(1)}D_{(1)}\right)c_{(0)}.$$

Let $\overline{\mathrm{OWP}}_n$ denote the collection of ordered weak set partitions of $[n]$ into $2n+1$ blocks, i.e., $[n] = B_0 \cup B_1 \cup \overline{B}_1 \cup B_2 \cup \overline{B}_2 \cdots \cup B_n \cup \overline{B}_n$, and for which the following conditions hold: (a) $1 \in B_0$; (b) if B_i or $\overline{B}_i$ is nonempty, then its minimum element larger than i. See Fig. 4 for illustrations.

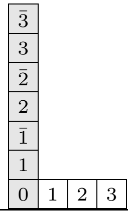
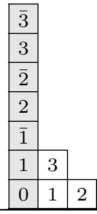
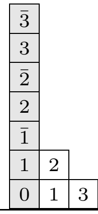
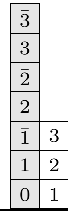
$T \in \mathrm{SYT}\,(3)$				
$\prod_{i=1}^{3} \delta_i(T)$	1	4	4	6
$w(T)$	c^6c_3	$c^5c_2c_1$	$c^5c_2c_1$	$c^4c_1^3$
$C_3(x)$	x	$4x^2$	$4x^2$	$6x^3$

Fig. 5. The illustrations of $(c^2D)^3\ c = c^6c_3 + 8c^5c_2c_1 + 6c^4c_1^3$ and $C_3(x) = x + 8x^2 + 6x^3$.

For any $p \in \overline{\mathrm{OWP}}_n$, the weight function of the corresponding standard Young tableau is defined by

$$w(T) = c^{2n+1-\ell(\lambda(T))} \prod_{i=1}^{n} c_i^{w_i(T)}. \tag{25}$$

For $T \in \mathrm{SYT}\,(n)$, recall that T_i is the element in $\mathrm{SYT}\,(i)$ obtained from T by deleting the $n - i$ elements $i + 1, i + 2, \ldots, n$.

Definition 30. The *second-order g-index* of the entry i in T is defined by

$$\delta_i(T) = \begin{cases} 2i - \mathrm{col}_1(T_i), & \text{if } i \text{ is in the first column;} \\ \mathrm{col}_k(T_i) - \mathrm{col}_{k+1}(T_i) + 1, & \text{if } i \text{ is in the } (k+1)\text{-th column, where } k \geqslant 1. \end{cases}$$

We call $\prod_{i=1}^{n} \delta_i(T)$ the *second-order g-index* of T.

In the same way as in the proof of Theorem 23, it is routine to check the following result, and we omit the proof for simplify.

Theorem 31. *For $n \geqslant 1$, we have $(c^2D)^n c = \sum_{p \in \overline{\mathrm{OWP}}_n} w(p)$ and*

$$(c^2D)^n c = \sum_{T \in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \delta_i(T) c_i^{w_i(T)} \right) c^{2n+1-\ell(\lambda(T))}. \tag{26}$$

We end this paper by giving the following result. See Fig. 5 for an illustration.

Theorem 32. *Let $C_n(x)$ be the second-order Eulerian polynomials. We have*

$$C_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \delta_i(T) \right) x^{\ell(\lambda(T))}.$$

Proof. From Example 9, we see that let $G = \{x \to x,\ y \to x\}$, then we have

$$(y^2 D_G)^n(y) = y^{2n+1} C_n\left(\frac{x}{y}\right).$$

Set $c = y$. Then $c_i = D_G^i(c) = D_G^i(y) = x$ for any $i \geqslant 1$. Using (26), we find that

$$C_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} \left(\prod_{i=1}^{n} \delta_i(T) c_i^{w_i(T)} \right) c^{2n+1-\ell(\lambda(T))} \Big|_{c=y,\ c_i=x,\ y=1},$$

which yields the desired result. This completes the proof. $\square$

Declaration of competing interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of the manuscript entitled.

Acknowledgments

We would like to thank the referees for carefully reading the paper and providing insightful comments and suggestions. The first author was supported by the National Natural Science Foundation of China (Grant number 12071063) and Taishan Scholars Program of Shandong Province (No. tsqn202211146). The third author was supported by the National Science and Technology Council (Grant number: MOST 1132115M017005MY2).

Data availability

No data was used for the research described in the article.

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